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Large deformations of elastic cylindrical capsules in shear flows

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The nonlinear problem of the steady-state interaction of a closed fluid-filled cylindrical elastic membrane with a slow viscous shear flow has been solved by a series-expansion technique. The problems of successive orders were both formulated and solved by a symbolic manipulation program, and the calculations were carried to sixth order in a dimensionless parameter related to the applied shear rate. Moderately large deformations (aspect ratios approaching 3) fall within the range of this analysis, which yields the dependences of the following global variables on the system parameters: membrane deformation, orientation, and strain, as well as tank-treading frequency, and mean internal pressure. The solution for the flow field around an isolated capsule is also used to calculate the apparent viscosity of a dilute suspension of flexible cylindrical particles, which yields the paradoxical result that the apparent viscosity decreases as the internal viscosity increases.

1. Introduction

We consider the deformations of thin closed elastic membranes filled with incompressible Newtonian liquids, when such bodies ('capsules') are placed in two-dimensional shear flows of another incompressible Newtonian liquid – under conditions when the induced deformations are large. This problem is motivated primarily by our interest in the flow of blood wherein a Newtonian fluid (the plasma) carries in suspension several kinds of capsules (the formed elements) including erythrocytes, leukocytes, and platelets. In particular, much attention has been focused in recent years on the deformability of erythrocytes (Schmid-Schönbein & Gaehtgens 1981) as this property is vital to their physiological function (Cranston *et al.* 1984). An assay of erythrocyte deformability has been devised employing an instrument called the Rheoscope: this is basically a counter-rotating cone-and-plate viscometer which subjects isolated erythrocytes to graded levels of fluid shear and measures the resulting cell deformations and motions (Fischer, Stöhr & Schmid-Schönbein 1978; Sutera *et al.* 1985). From Rheoscope measurements it is possible to calculate the viscoelastic characteristics of the erythrocyte membrane, but this requires mathematical modelling and analysis so that the global experimental measurements can be interpreted in terms of local cellular properties (Tran-Son-Tay, Sutera & Rao 1984; Tran-Son-Tay *et al.* 1987; Sutera, Pierre & Zahalak 1989). The objective of the work reported in this paper was to carry out a detailed mathematical investigation of a simplified model resembling an erythrocyte in the Rheoscope, in the expectation that the results obtained would provide some insight into the many complex interactions which govern the shear-induced motion and deformation of an erythrocyte. In addition, of course, the problem of flows containing flexible particles is one of general interest in suspension rheology.

The major difficulty in constructing an accurate model of sheared erythrocyte suspensions is that these particles are very deformable indeed (Fischer & Schmid-Schönbein 1977). Consequently, no complete analysis of this problem – which calculates grossly deformed capsule shape from the undeformed shape, membrane properties, and shear flow intensities – has yet been published. A number of approximate analyses, both computational and analytical, have appeared in the literature; we have reviewed these studies in a previous paper on this topic (Zahalak, Rao & Sutero 1987) and will not repeat this survey here. Let it suffice to say that all previous analyses either (i) were limited to small deformations of the capsules, or (ii) assumed *a priori* the deformed shape and surface velocity field of the deformed capsule. The prior work most closely related to our present investigation is that of Barthès-Biesel and her collaborators (Barthès-Biesel & Rallison 1981; Barthès-Biesel & Sgaier 1985) who have applied perturbation analysis to calculate the deformations of initially spherical viscoelastic capsules in shear flows. Their work is an extension of Taylor's classic analysis of liquid droplets (Taylor 1932), and like Taylor's work it is limited to small capsule deformations. Recently, analyses of genuinely large deformations of fluid-filled capsules have been carried out using the technique of boundary-integral equations (Li, Barthès-Biesel & Helmy 1988; Pozrikidis 1990). This approach seems to have great potential but (with one exception) all the problems solved with it so far have assumed rotational symmetry, which does not occur in the case of an erythrocyte embedded in a shear flow. (The one exception is a very recent abstract (Uijttewaal, Nijhal & Heethaar 1992) describing the boundary integral solution of the problem of a liquid drop near a rigid boundary.)

We have undertaken an extensive analytical study of a simplified model of a viscoelastic capsule in a slow viscous shear flow, which is described in detail in Rao (1990, 1991). Although we considered both transient and steady-state behaviour, only the latter will be discussed in this paper. Our study treated a thin viscoelastic membrane with the undeformed shape of a long circular cylinder, which is filled with a viscous Newtonian fluid and placed in a shear flow of another Newtonian fluid; the axis of the cylinder is perpendicular to the plane of the flow. The membrane was assumed to be of an area-conserving Kelvin-type viscoelastic material which has been proposed as a model for the erythrocyte membrane (Evans & Skalak 1980). The major simplifying assumption of our analysis was that the flow inside and outside the membrane was two-dimensional. Consistent with this, we assumed further that the axial strain in the membrane was constrained to be independent of position. When combined with the area-conservation assumption this implies that the circumferential, as well as the axial, membrane strain can vary with time but not with position on the membrane. An immediate consequence of these assumptions is that the membrane strain rate is zero in a steady state. That is, the viscous membrane stresses vanish, and the membrane behaves as though it were made of a purely elastic material. Therefore, although the analysis to be presented in this paper is embedded in a more general analysis, we will restrict discussion to elastic membranes. No *a priori* assumptions were made about the shape or velocity of the deformed membrane cross-section and the results are expected to be at least qualitatively valid indications of the complex relations between membrane deformation, stress, internal pressure, shear rate, fluid viscosities, and membrane elasticity.

Our approach was to use a combination of perturbation analysis and computer algebra. As in several previous studies, the field variables of the problem were expanded in asymptotic power series of a dimensionless shear-rate parameter. The restriction to small deformations was removed by carrying out the expansions to relatively high order yielding four non-zero terms for each field variable, and subsequently applying conformal transformation to the solution. In order to handle the massive algebraic manipulations inherent in this approach we employed a symbolic manipulation program (REDUCE 3.1, Rand Corporation, Santa Monica, CA, 1983) which both formulated the problems for the higher-order terms and solved them. We have previously published a report (Zahalak *et al.* 1987) of a similar investigation which applied this technique to a simpler problem than the one considered here, namely deformations of an inextensible, initially pressurized cylindrical membrane by a viscous shear flow. In this paper, we deal with an elastic membrane which is initially unpressurized. The basic technique of analysis, however, is the same as in our earlier paper and the reader may refer to the latter for many of the details. As a particular application of these results we calculated the apparent viscosity of a dilute suspension of flexible capsules, and how this viscosity varies with the parameters of the system.

2. Formulation

Figure 1 defines the geometry of the problem. Cartesian coordinates $(\hat{x}, \hat{y})$ and polar coordinates $(\hat{r}, \theta)$ are established in the plane of an undisturbed shear flow defined by $\hat{u}_{\hat{x}} = \dot{\gamma}\hat{y}$, where $\dot{\gamma}$ is a shear rate and $\hat{u}_{\hat{x}}$ is the horizontal component of the fluid velocity. Unit vectors associated with these coordinate systems are, respectively, $(\boldsymbol{e}_{\hat{x}}, \boldsymbol{e}_{\hat{y}})$ and $(\boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\theta})$. (Throughout this paper superposed carets will denote physical variables, and the absence of carets will denote corresponding dimensionless variables; the single exception is $\dot{\gamma}$, which denotes the physical shear rate.) A cylindrical capsule consisting of a thin membrane filled with an incompressible Newtonian liquid of viscosity ${}^{i}\hat{\mu}$ is placed in the shear flow with its axis perpendicular to the plane of flow and lying in the zero-velocity plane. When unloaded the cross-section of the capsule is circular with radius $\hat{a}$. The presence of the capsule will disturb the shear flow and, in turn, stresses exerted by the shear flow will deform the capsule into the quasi-elliptical shape shown in figure 1. Unit vectors tangent and normal to the deformed membrane are defined as $\boldsymbol{e}_{\hat{s}}$ and $\boldsymbol{e}_{\hat{n}}$ (the former is in the counterclockwise direction while the latter points into the capsule).

To describe the mechanical properties of the membrane, we will adopt the well known model of Evans & Skalak (1980) for an elastic membrane. Let λ_1 denote the stretch ratio of a membrane element in the circumferential direction (that is, in the direction of $\boldsymbol{e}_{\hat{s}}$) and λ_2 the stretch ratio in the axial direction, perpendicular to the plane of flow. The membrane is assumed to be two-dimensionally incompressible (i.e. area-conserving) which implies that $\lambda_1 = \lambda_2^{-1}$. Both λ_1 and λ_2 are assumed to be principal stretch ratios, and if $\hat{P}$ and $\hat{Q}$ represent the corresponding membrane tensions (force per unit length) the constitutive equations of the membrane material are

$$\hat{P} = \hat{T} + \tfrac{1}{2}\hat{G}(\lambda^2 - \lambda^{-2}), \tag{1}$$

$$\hat{Q} = \hat{T} - \tfrac{1}{2}\hat{G}(\lambda^2 - \lambda^{-2}), \tag{2}$$

or

$$\hat{P} - \hat{Q} = \hat{G}(\lambda^2 - \lambda^{-2}), \tag{3}$$

where $\lambda = \lambda_1$, $\hat{T}$ is an isotropic surface tension, and $\hat{G}$ is the surface shear modulus of elasticity (force/length).

The mass of the membrane is assumed to be negligibly small so that each element of the membrane must be in equilibrium under the action of membrane tensions and the pressures and viscous stresses which are exerted by the external and internal fluids. As

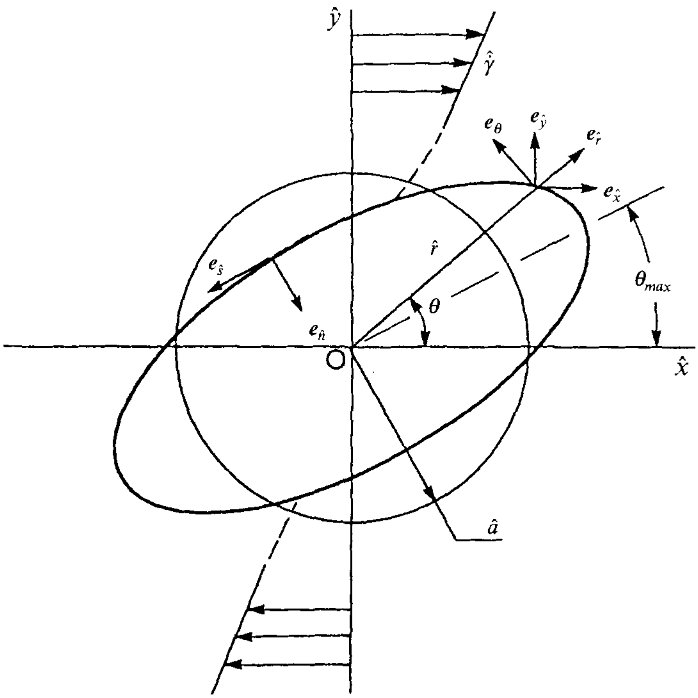

FIGURE 1. Flow geometry. Cartesian and cylindrical polar coordinates are denoted by $\hat{x}, \hat{y}$, and $\hat{r}, \theta$, respectively. Arclength along the membrane is $\hat{s}$ (positive in the counterclockwise direction), and distance normal to the membrane is $\hat{n}$ (positive in the inward direction); e denotes a unit base vector in the direction indicated by the subscript. The axis of the cylinder of initial radius $\hat{a}$ is contained in the plane of zero velocity ($\hat{y} = 0$), and the deformed shape is described by $\hat{r} = \hat{f}(\theta)$, where $\hat{f}$ is the shape function.

noted previously, inertial effects are considered negligible, and therefore the internal and external fluids are governed by the equations of creeping (Stokes) flow. Further, as the flow is both incompressible and two-dimensional, the internal and external velocity fields can be expressed in terms of scalar internal and external stream functions, ${}^i\hat{\psi}$ and ${}^e\hat{\psi}$, respectively. (Throughout this paper the superscripts i and e placed before a variable refer, respectively, to the interior and exterior of the capsule.) The formulation of this problem follows closely that which we have published previously for an inextensible membrane (Zahalak *et al.* 1987) and therefore we will list the final equations directly. We first define dimensionless variables as follows: all lengths are normalized by the undeformed capsule radius $\hat{a}$, pressures and viscous stresses by $\hat{G}/\hat{a}$, membrane tensions by $\hat{G}$, stream functions by $\hat{a}\hat{G}/{}^e\hat{\mu}$, and velocities by $\hat{G}/{}^e\hat{\mu}$. There are two important dimensionless parameters in this problem: the dimensionless applied shear rate $\epsilon = \hat{a}\dot{\gamma}\,{}^e\hat{\mu}/\hat{G}$ and the viscosity ratio $\beta = {}^i\hat{\mu}/{}^e\hat{\mu}$. In terms of these dimensionless variables the problem to be solved is as follows.

Find a simple, closed curve Γ: $r = f(\theta)$ (the capsule shape), internal stream function ${}^i\psi$ and pressure ${}^ip$, external stream function ${}^e\psi$ and pressure ${}^ep$, membrane tensions P and Q, and membrane stretch ratio λ satisfying the following equations and boundary conditions:

$$\nabla^4\,{}^e\psi = 0 \quad \text{and} \quad \nabla\times\nabla^2\,{}^e\psi e_z = \nabla\,{}^ep \quad \text{outside} \quad \Gamma, \tag{4}$$

$$\nabla^4\,{}^i\psi = 0 \quad \text{and} \quad \beta\nabla\times\nabla^2\,{}^i\psi e_z = \nabla\,{}^ip \quad \text{inside} \quad \Gamma, \tag{5}$$

$${}^e\psi \rightarrow \epsilon r^2(1-\cos 2\theta)/4 \quad \text{and} \quad {}^ep \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty. \tag{6}$$

On Γ:

$$ {}^{i}\psi_{,\theta} = {}^{e}\psi_{,\theta}, \tag{7} $$

$$ {}^{i}\psi_{,r} = {}^{e}\psi_{,r}, \tag{8} $$

$$ {}^{e}\psi = 0, \tag{9} $$

$$ {}^{e}u_s = {}^{e}u_s(\epsilon, \beta), \tag{10} $$

$$ \mathrm{d}P/\mathrm{d}s = (\boldsymbol{e}_n \cdot {}^{e}\boldsymbol{\tau} \cdot \boldsymbol{e}_s) - \beta(\boldsymbol{e}_n \cdot {}^{i}\boldsymbol{\tau} \cdot \boldsymbol{e}_s), \tag{11} $$

$$ \kappa P = {}^{i}p - {}^{e}p, \tag{12} $$

$$ P - Q = \lambda^2 - \lambda^{-2}. \tag{13} $$

Further, we impose a 'global equilibrium condition' of the form

$$ \oint_{\Gamma} Q\,\mathrm{d}s = \iint_{\mathscr{A}} {}^{i}p\,\mathrm{d}\mathscr{A}, \tag{14} $$

and require also that ${}^{i}\psi$ have no singularities inside the membrane.

In these equations the comma notation is employed to indicate partial differentiation (e.g. ${}^{i}\psi_{,r} = \partial\, {}^{i}\psi/\partial r$). Also,

$$ \kappa = \rho^{-1} = (f^2 + 2f_{,\theta}^2 - ff_{,\theta\theta})/(f^2 + f_{,\theta}^2)^{\frac{3}{2}} \tag{15} $$

is the membrane curvature, and $p, \boldsymbol{\tau}$ and $\mathrm{d}s$ represent, respectively, the pressure, fluid viscous stress tensor, and membrane arclength. Finally,

$$ {}^{e}u_s = ({}^{e}\psi_{,\theta} f_{,\theta} - {}^{e}\psi_{,r} f^2)/(f(f^2 + f_{,\theta}^2)^{\frac{1}{2}}) \tag{16} $$

is the tangential external velocity component on the membrane. The viscous shear stress components appearing in (11) can be expressed in terms of $f, {}^{i}\psi$ and ${}^{e}\psi$; these expressions are given as equation (19) of Zahalak *et al.* (1987).

Some comments on the significance of these equations and boundary conditions, (4)–(14), are in order. Equations (4) and (5) are simply the equations of creeping flow which govern the motions of the internal and external fluids, written in terms of the stream functions and pressures. The first of the limit statements (6) says that the velocity field far from the capsule becomes a simple shear flow, whereas the second states that pressures are measured with respect to the constant remote pressure. Equations (7)–(10) are kinematic boundary conditions; the first and second of these assert continuity of the fluid velocity across the membrane, and (9) expresses the condition that in the steady state the membrane is a streamline. Equation (10) follows from the assumption that λ is independent of position, and asserts that the membrane and fluid speeds are constant on the membrane (see Tran-Son-Tay *et al.* 1987). (Note that since the internal and external fluid velocities are equal on the membrane by virtue of (7) and (8), the superscript e could be replaced by i in (10) and (16).) Equations (11) and (12) express the equilibrium of forces acting on the membrane. The first of these is for the direction tangent to the membrane, and the second is for the normal direction. It can be shown that the normal viscous stresses of the external and the internal fluids make no contribution to the second equilibrium equation, as a consequence of the fact that the membrane speed is constant along the membrane. Equation (13) is simply the dimensionless form of the constitutive relation, (3). Equation (14) is a global equilibrium condition which states that the axial force due to the internal pressure acting on hypothetical end-caps of the capsule is balanced by the axial membrane tension acting around the circumference of the capsule. This boundary condition neglects the axial force on the hypothetical end-caps due to the external fluid pressure, and some justification for this assumption will be provided in §6. Finally, we impose a condition of global volume conservation for the fluid inside the capsule in the form

$$\mathscr{V} = \mathscr{V}_u, \quad \text{so} \quad \mathscr{A} = \mathscr{A}_u(\mathscr{L}_u/\mathscr{L}) \quad \text{or} \quad \mathscr{A} = \pi\lambda.$$

So, finally

$$\overline{f^2} = \lambda, \tag{17}$$

where $\mathscr{V}$ and $\mathscr{A}$ represent the volume and the cross-sectional area of the capsule, respectively, the subscript u refers to the undeformed state, and the overbar denotes the angular average of a quantity, $(\overline{*}) = (1/2\pi)\int_0^{2\pi}(*)\,\mathrm{d}\theta$.†

3. Series expansions

The problem we have formulated is a difficult one in which complicated boundary conditions must be imposed on an *a priori* unknown boundary. As in our previous study of a pressurized, inextensible membrane (Zahalak *et al.* 1987), we will construct the solution by expanding all field variables in asymptotic series of a gage function $\delta(\epsilon)$. In contrast to the previous problem however, the forms of the expansions (including the form of the gage function) in the current problem are not obvious. There is, of course, no mechanical procedure for determining the correct forms of the expansions; one must assume a form, insert this form into the equations and boundary conditions, and if this leads to a sequence of solvable, well-posed problems on the terms of the expansion, then one assumes *a posteriori* that the postulated form is correct. Consideration of various expansions led us to the following:

$$f = 1 + \delta f^{(1)} + \delta^2 f^{(2)} + \delta^3 f^{(3)} + \ldots, \tag{18}$$

$${}^e\psi = \delta^3\, {}^e\psi^{(3)} + \delta^4\, {}^e\psi^{(4)} + \ldots, \tag{19}$$

$${}^i\psi = \delta^3\, {}^i\psi^{(3)} + \delta^4\, {}^i\psi^{(4)} + \ldots, \tag{20}$$

$${}^e p = \delta^3\, {}^e p^{(3)} + \delta^4\, {}^e p^{(4)} + \ldots, \tag{21}$$

$${}^i p = \delta^2\, {}^i p^{(2)} + \delta^3\, {}^i p^{(3)} + \delta^4\, {}^i p^{(4)} + \ldots, \tag{22}$$

$$P = \delta^2 P^{(2)} + \delta^3 P^{(3)} + \delta^4 P^{(4)} + \ldots, \tag{23}$$

$$Q = \delta^2 Q^{(2)} + \delta^3 Q^{(3)} + \delta^4 Q^{(4)} + \ldots, \tag{24}$$

$$\lambda = 1 + \delta^2 \lambda^{(2)} + \delta^4 \lambda^{(4)} + \ldots, \tag{25}$$

where $\delta = \epsilon^{\frac{1}{3}}$. It is implied that the leading terms of the right-hand sides of (18)–(25) are non-zero. Expansions of this form permitted the construction of a consistent recursive algorithm through which terms of any order could be calculated, and the solution has been carried out up to terms of $O(\delta^4)$ in the shape function and $O(\delta^6)$ in the stream functions. To carry out the solution beyond the lowest terms requires a formidable amount of algebra, so the procedure was programmed in REDUCE, a symbolic manipulation program, which both formulated the successive problems and solved them. The algorithm is straightforward but lengthy; specific details and the REDUCE program listing are available in a technical report by Rao (1990).

Space limitations preclude any detailed discussion of the solution algorithm, which was essentially similar to that described in Zahalak *et al.* (1987). In this section we will merely point out a few salient features of the analysis; a more detailed exposition of the solution can be found in Rao (1990, 1991). The solution begins by assuming that the membrane shape function of order m can be written as a trigonometric series

$$f^{(m)} = C_{m0} + \sum_{n=1}^{\infty} (S_{mn} \sin n\theta + C_{mn} \cos n\theta), \tag{26}$$

† In Rao (1990, 1991) the problem posed in (4)–(14) was solved with the auxiliary condition $(\overline{f^2}) = 1$ instead of (17). These results are easily converted to those presented in this paper by replacing $\hat{a}$ in Rao (1990, 1991) by $\lambda^{\frac{1}{2}}\hat{a}$.

and that both the external and internal stream functions can be written in the form

$$\psi^{(m)} = A_{10} + \sum_{n=1}^{\infty} (A_{1n} r^{-n} \cos n\theta + A_{2n} r^n \cos n\theta + B_{1n} r^{-n} \sin n\theta + B_{2n} r^n \sin n\theta)$$
$$+ C_1 \log r + (r^2-1)\left[A_{30} + \sum_{n=1}^{\infty} \{A_{3n} r^{-n} \cos n\theta + A_{4n} r^n \cos n\theta\right.$$
$$\left. + B_{3n} r^{-n} \sin n\theta + B_{4n} r^n \sin n\theta\} + C_2 \log r\right]. \tag{27}$$

Equation (27) is a general solution of the biharmonic equation satisfied by the stream functions (Duff & Naylor 1966, p. 143), and the logarithmic terms therein can be deleted on the basis of the imposed boundary conditions (Zahalak *et al.* 1987). The coefficients of the trigonometric terms in (26) and (27), must be determined to solve the problem. These assumed forms of shape and stream functions are inserted into the differential equations and the boundary conditions of the problem ((4)–(17)) and the coefficients of like powers of δ are extracted to yield a sequence of perturbation problems of increasing order. In this process the boundary conditions on the unknown boundary $r = f(\theta)$ are expanded in Taylor series and applied on the known fixed boundary $r = 1$, as in Zahalak *et al.* (1987).

The specific solution strategy involves expressing all field variables in terms of the coefficients of the shape functions, S_{mn} and C_{mn}, then establishing equations satisfied by these coefficients, using the differential equations and the boundary conditions of the original problem, and finally solving for S_{mn} and C_{mn} to define all the field variables. In this process it becomes clear that there are only a finite number of non-zero trigonometric terms in the solution for each field variable at any order. An interesting consequence of the fact that the expansions ((18)–(25)) of different field variables begin with leading terms of different orders is that in order to completely determine the solution of order m the problem of order $m+3$ must be examined. In particular, to determine the leading term of the shape function the problems of order δ to δ^4 must be examined. The details are explained in Rao (1990, 1991).

In this manner we were able to obtain the following forms of expansions for all the field variables up to third order in δ:

$$O(\delta):\quad f^{(1)} = S_{12} \sin 2\theta, \tag{28}$$
$$O(\delta^2):\quad f^{(2)} = C_{22} \cos 2\theta + C_{24} \cos 4\theta + \tfrac{1}{8} S_{12}^2, \tag{29}$$
$${}^{i}p^{(2)} = 6S_{12}^2, \tag{30}$$
$$P^{(2)} = 6S_{12}^2, \tag{31}$$
$$Q^{(2)} = 3S_{12}^2, \tag{32}$$
$$\lambda^{(2)} = \tfrac{3}{4} S_{12}^2; \tag{33}$$
$$O(\delta^3):\quad f^{(3)} = S_{32} \sin 2\theta + S_{34} \sin 4\theta + S_{36} \sin 6\theta, \tag{34}$$
$${}^{e}\psi^{(3)} = \tfrac{1}{4}(r^2-1) + \cos 2\theta(\tfrac{1}{2} - \tfrac{1}{4}r^2 - \tfrac{1}{4}r^{-2}), \tag{35}$$
$${}^{i}\psi^{(3)} = \tfrac{1}{4}(r^2-1), \tag{36}$$
$${}^{e}p^{(3)} = -2r^{-2} \sin 2\theta, \tag{37}$$
$${}^{i}p^{(3)} = 0, \tag{38}$$
$$P^{(3)} = -\sin 2\theta, \tag{39}$$
$$Q^{(3)} = -\sin 2\theta. \tag{40}$$

The coefficients S_{ij} and C_{ij} appearing in the above expressions are constants depending on β. Specifically,

$$\left.\begin{aligned} S_{12} = 6^{-\frac{1}{3}}, \quad C_{22} = 6^{-\frac{2}{3}}(1+\beta), \quad C_{24} = -\tfrac{1}{8}(6^{\frac{1}{3}}), \\ S_{32} = 6^{-\frac{2}{3}}\tfrac{7}{4} - 13/288 - \tfrac{1}{9}(1+\beta)^2, \quad S_{34} = \tfrac{1}{4}(1+\beta), \quad S_{36} = -5/48. \end{aligned}\right\} \tag{41}$$

As in our previous analysis of the pressurized, inextensible membrane, we employed in this study a simple conformal transformation which maps to first order the region outside the deformed capsule onto the region outside the unit circle (see Zahalak *et al.* 1987, §5). This enabled us to improve the accuracy of most of the expansions we had generated in the original physical variables r and θ by straightforward algebraic manipulations not requiring the solution of any additional boundary value problems. The results which follow were computed using conformally transformed expressions of our original solutions (see Rao 1990 for details).

4. Results

Although the REDUCE computer program yielded 'exact' algebraic expressions for each term in the expansions of the field variables, the complexity of these expressions grew extremely rapidly with order, to the point where it is impractical to display the solution beyond $O(\delta^3)$. With such complex results what assurance do we have that they are correct? It is obvious from the forms of the stream functions that the differential equations are indeed satisfied, so the question of validity devolves to the boundary conditions. We have used a computational test, described in Zahalak *et al.* (1987), to demonstrate in several distinct cases that the error in the boundary conditions is $O(\delta^7)$ as δ approaches zero. (See figure 2 of the cited reference, and figure 5.35 of Rao 1991.) This demonstrates that the boundary conditions are satisfied up to $O(\delta^6)$, and on this basis we accept our expansions as correct to this order.

Another question concerns the range of validity of the expansions: How large can δ be if we are to be assured of adequate accuracy? This is a difficult question, the answer to which depends on the accuracy criterion, the variable being evaluated, and the values of the system parameters. Having available four-term expansions (from solutions to problems up to the sixth-order in δ) we have been able to examine the question of accuracy in some detail (Rao 1991), but an adequate discussion of our investigations would take us too far afield. We will qualitatively summarize our conclusions by stating that for accuracy (i) ϵ should be less than about 0.25 at $\beta = 2$, and less than about 0.5 up to $\beta \approx 1$, and (ii) the field variables listed in order of decreasing accuracy are as follows: $f, P, {}^e p, \boldsymbol{e}_n \cdot {}^e\boldsymbol{\tau} \cdot \boldsymbol{e}_s, {}^i p, \boldsymbol{e}_n \cdot {}^i\boldsymbol{\tau} \cdot \boldsymbol{e}_s$. Thus, for example, for a given set of parameters and desired accuracy, the maximum allowable value of ϵ (or δ) will be less for the shear stress on the membrane than that for the membrane shape. In general, it may be assumed that the results displayed in this paper are correct within graphical accuracy. In particular, our results describe the deformed membrane shape up to aspect ratios of 2.6 when $\beta = 0$.

The solution up to $O(\delta^3)$ was presented earlier in (28)–(41). An analysis of the highest-order terms is presented in Rao (1990). Tables 1 and 2 list, respectively, the resulting terms of the external and the internal stream functions up to $O(\delta^6)$. Note that the complex Goursat representation (Zahalak *et al.* 1987) is used to represent both of these functions. The steady-state variations of the 'local' variables (velocity, pressure, fluid stresses on the membrane, membrane tension) were found to be qualitatively similar to those reported for the pressurized capsule in Zahalak *et al.* (1987). Indeed, it can be shown that *to first-order* the solutions for these local variables in the present

	$\downarrow n;\, m \rightarrow$	3	4	5	6
A_{mn}	2	$-\frac{1}{4}$	0	$-\frac{3}{32}S_{12}^2$	0
	0	$-\frac{1}{4}$	0	$\frac{5}{8}+\frac{3}{32}S_{12}^2$	0
	-2	$-\frac{1}{4}$	$\frac{1}{4}jS_{12}$	$\frac{1}{4}\{-\frac{37}{8}S_{12}^2+C_{22}-2C_{24}\}$	$\frac{1}{4}j\{-\frac{41}{8}S_{12}^3-13S_{12}C_{24}+3S_{12}C_{22}+S_{32}-2S_{34}\}$
	-4	0	$-\frac{1}{2}jS_{12}$	$\frac{1}{4}\{-2C_{22}+3C_{24}\}$	$\frac{1}{4}j\{-\frac{17}{4}S_{12}^3+8S_{12}C_{24}-2S_{32}+3S_{34}-2S_{36}\}$
	-6	0	0	$\frac{1}{4}\{\frac{7}{2}S_{12}^2-C_{24}\}$	$\frac{1}{4}j\{-\frac{3}{4}S_{12}^3+3S_{12}C_{24}-7S_{12}C_{22}-2S_{34}+5S_{36}\}$
	-8	0	0	0	$\frac{1}{4}j\{6S_{12}^3-9S_{12}C_{24}-2S_{36}\}$
B_{mn}	1	$\frac{1}{4}$	0	$\frac{3}{32}S_{12}^2$	0
	-1	$\frac{1}{2}$	$-\frac{3}{4}jS_{12}$	$\frac{1}{4}\{\frac{17}{4}S_{12}^2-3C_{22}+2C_{24}\}$	$\frac{1}{4}j\{\frac{21}{8}S_{12}^3+12S_{12}C_{24}-S_{12}C_{22}-3S_{32}+2S_{34}\}$
	-3	0	$\frac{1}{2}jS_{12}$	$\frac{1}{4}\{-\frac{1}{2}S_{12}^2+2C_{22}-5C_{24}\}$	$\frac{1}{4}j\{\frac{1}{4}S_{12}^3+S_{12}C_{22}-8S_{12}C_{24}+2S_{32}-5S_{34}+2S_{36}\}$
	-5	0	0	$\frac{1}{4}\{-\frac{5}{2}S_{12}^2+2C_{24}\}$	$\frac{1}{4}j\{S_{12}^3-2S_{12}C_{24}+5S_{12}C_{22}+2S_{34}-7S_{36}\}$
	-7	0	0	0	$\frac{1}{4}j\{-\frac{7}{2}S_{12}^3+7S_{12}C_{24}+2S_{36}\}$

TABLE 1. Coefficients A_{mn} and B_{mn} in the external stream function

$$^{e}\psi = \mathrm{Re}\sum_{m=3}^{\infty}\delta^m\{^{e}\chi_m(z)+\bar{z}\,^{e}\Phi_m(z)\},$$

where

$$^{e}\chi_m(z) = \sum_{n=-\infty}^{\infty}A_{mn}z^n \text{ and } ^{e}\Phi_m(z) = \sum_{n=-\infty}^{\infty}B_{mn}z^n.$$

Note that A_{mn} and B_{mn} are completely determined by S_{ij} and C_{ij}, the coefficients in the membrane shape function which are given by (41). These relations between A_{mn}, B_{mn} and S_{ij}, C_{ij} are independent of the material parameters.

	$\downarrow n;\, m \rightarrow$	3	4	5	6
A_{mn}	0	$-\frac{1}{4}$	0	$-\frac{3}{32}S_{12}^2+\frac{1}{2}C_{22}$	0
	2	0	$\frac{3}{4}jS_{12}$	$-\frac{3}{4}C_{22}$	$\frac{1}{4}j\{\frac{57}{8}S_{12}^3+15S_{12}C_{24}-6S_{12}C_{22}+3S_{32}\}$
	4	0	0	$\frac{1}{4}\{-\frac{1}{2}S_{12}^2-5C_{24}\}$	$\frac{1}{4}j\{-4S_{12}C_{22}+5S_{34}\}$
	6	0	0	0	$\frac{1}{4}j\{-\frac{15}{4}S_{12}^3-9S_{12}C_{24}+7S_{36}\}$
B_{mn}	1	$\frac{1}{4}$	0	$\frac{3}{32}S_{12}^2-\frac{1}{2}C_{22}$	0
	3	0	$-\frac{1}{4}jS_{12}$	$\frac{1}{4}C_{22}$	$\frac{1}{4}j\{-\frac{37}{8}S_{12}^3-14S_{12}C_{24}+2S_{12}C_{22}-S_{32}\}$
	5	0	0	$\frac{1}{4}\{\frac{3}{2}S_{12}^2+3C_{24}\}$	$\frac{1}{4}j\{3S_{12}C_{22}-3S_{34}\}$
	7	0	0	0	$\frac{1}{4}j\{\frac{7}{2}S_{12}^3+8S_{12}C_{24}-5S_{36}\}$

TABLE 2. Coefficients A_{mn} and B_{mn} in the internal stream function

$$^{i}\psi = \mathrm{Re}\sum_{m=3}^{\infty}\delta^m\{^{i}\chi_m(z)+\bar{z}\,^{i}\Phi_m(z)\},$$

where

$$^{i}\chi_m(z) = \sum_{n=-\infty}^{\infty}A_{mn}z^n \text{ and } ^{i}\Phi_m(z) = \sum_{n=-\infty}^{\infty}B_{mn}z^n.$$

See note in table 1.

paper can be obtained from those given in Zahalak *et al.* (1987) by simply introducing the parameter transformation $\bar{\epsilon} = 6^{-\frac{1}{3}}\delta$, where $\bar{\epsilon} = (\dot{\gamma}\,^{e}\mu/p_0)$ is the dimensionless shear rate of the latter paper; this is a direct consequence of the fact that in the present solution the first-order variation of the internal pressure was found to be $6^{\frac{1}{3}}\delta^2$. Consequently, with one exception, we will not provide displays of the local variables, which may in any case be found in Rao (1991). The exception is the circumferential membrane tension, P, which is exhibited in figure 2, and has a somewhat different behaviour from the corresponding variable in the earlier problem of the initially pressurized membrane. For every ϵ and β the membrane experiences no compressive stress. The minimum tension occurs close to the region of maximum curvature and *vice*

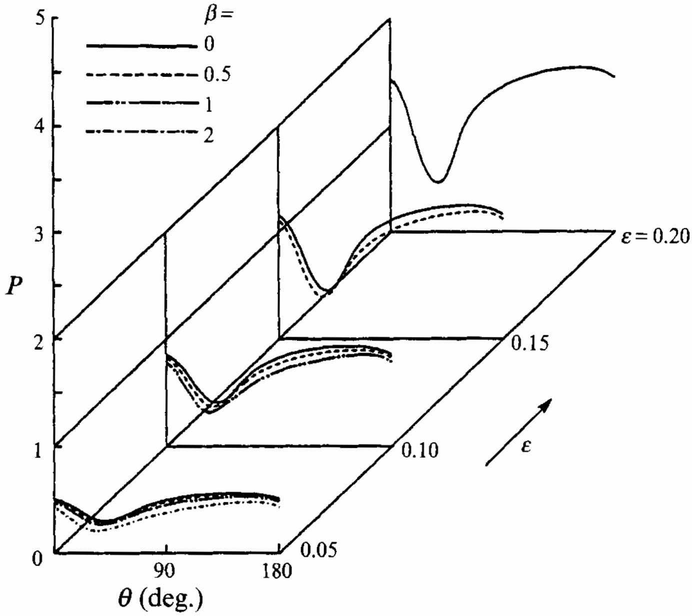

FIGURE 2. Membrane circumferential tension, as a function of ϵ and β.

versa. The mean tension increases strongly with increasing ϵ and weakly with decreasing β, and is never negative (i.e. compressive).

Five 'global' variables which characterize the overall deformation of the membrane are displayed in figure 3. These are the aspect ratio, the orientation, the tank-treading frequency, the mean internal pressure, and the circumferential stretch ratio. The membrane deformation is described by its aspect ratio, $D = (r_{max}/r_{min})$ in figure 3(*a*), which shows the dependence of this variable on ϵ and β. The deformation increases with ϵ (that is, with the applied shear rate, the external viscosity, and the inverse of the membrane shear modulus), rapidly at lower values of ϵ and more gradually at higher values of ϵ. The deformation decreases slightly as β increases. Coincident with an increased deformation at large ϵ is a decrease in the angle of inclination of the membrane, θ_{max} (see figure 1), with respect to the shear direction. Figure 3(*b*) shows this angle as a function of ϵ and β. The membrane approaches an orientation of 45° as $\epsilon \rightarrow 0$ for all β, and this angle decreases as either ϵ or β increases. In contrast to its effect on the shape, the viscosity ratio has a relatively strong effect on capsule orientation.

The tank-treading frequency ($\hat{\omega}$) is the rate of rotation of the membrane around its interior at steady state. It is related to the tangential membrane velocity as follows. If $\hat{\omega}$, ${}^m\hat{u}_{\hat{s}}$, and $\hat{\mathscr{C}}$ are, respectively, the dimensional tank-treading frequency in rad s^{-1}, tangential membrane velocity, and the membrane circumference, then

$$\hat{\omega} = {}^m\hat{u}_{\hat{s}} 2\pi/\hat{\mathscr{C}}, \tag{42}$$

$$= {}^m u_s \frac{\hat{G}}{{}^e\hat{\mu}} \frac{2\pi}{2\pi \hat{a} \lambda},$$

$$= \frac{{}^m u_s}{\lambda} \frac{\hat{G}}{\hat{a}\,{}^e\hat{\mu}}. \tag{43}$$

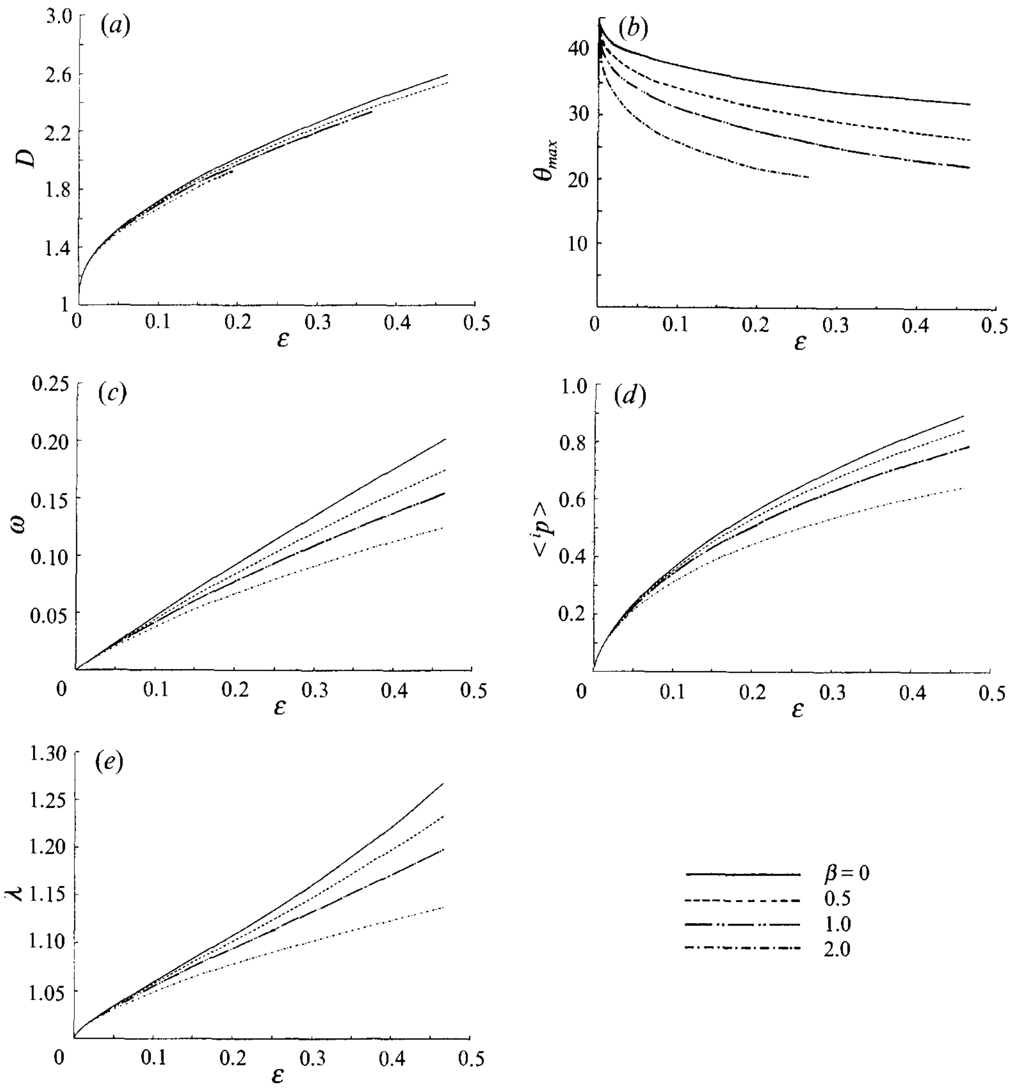

FIGURE 3. Global Variables. The influence of ϵ and β on (*a*) the membrane deformation, D (the aspect ratio); (*b*) angle of inclination, θ_{max}; (*c*) tank-treading frequency, ω; (*d*) mean (area-averaged) internal pressure, $\langle {}^ip \rangle$; and (*e*) circumferential stretch ratio, λ, are illustrated for $\beta = 0$, 0.5, 1 and 2. In all cases the curve for higher β is lower. For a given β as ϵ increases all the variables shown increase, except θ_{max} which decreases.

By non-dimensionalizing $\hat{\omega}$ by $\hat{G}/\hat{a}^e\hat{\mu}$ we obtain the dimensionless tank-treading frequency,

$$\omega = {}^mu_s/\lambda. \tag{44}$$

The membrane velocity at steady state can be computed from the stream function ${}^i\psi$ as

$$|{}^mu_s| = ({}^mu_r^2 + {}^mu_\theta^2)^{\frac{1}{2}} = \{({}^i\psi_{,\theta}/r)^2_{r=f(\theta)} + (-{}^i\psi_{,r})^2_{r=f(\theta)}\}^{\frac{1}{2}}. \tag{45}$$

Equivalently, the membrane speed can be computed from the exterior stream function, ${}^e\psi$, and is independent of θ, as indicated by (10). Although the solution for ω is carried out to sixth-order, the even terms (in δ) of the series turn out to be zero, as they must be from symmetry considerations. Consequently, only three non-zero terms of ω are known. In order to extend the range of validity of this limited series for larger values of shear rate, we apply the well-known technique of Padé's approximants (Bender & Orszag 1978). It has been demonstrated that Padé's approximants, for certain infinite series whose first few terms are known, yield a good approximation to the sum of the series even if the sum of the known terms of the series is a poor approximation. Padé's approximants $\mathscr{P}(f)$ to series f whose first two or three terms are known are defined as follows. If $f = a_1\delta + a_2\delta^2$, then $\mathscr{P}(f) = (a_1^2\delta)/(a_1 - a_2\delta)$, and if $f = a_0 + a_1\delta + a_2\delta^2$, then $\mathscr{P}(f) = (a_0 a_1 + (a_1^2 - a_0 a_2)\delta)/(a_1 - a_2\delta)$. We applied Padé's approximation to the tank-treading frequency (ω), and also the mean internal pressure, $\langle {}^i p \rangle$ and the circumferential stretch ratio, λ. Owing to symmetry restrictions $\langle {}^i p \rangle$ and λ contain only terms even in δ, as was the case for ω. In the remainder of the paper when we refer to values of any of the above three variables we mean those of their Padé's approximants.

The variation of ω with ϵ and β is displayed in figure 3(*c*). At constant β, ω increases monotonically with ϵ, and indeed is almost proportional to ϵ when $\beta = 0$. At constant ϵ, increasing β decreases ω, as one might expect intuitively. Figure 3(*d*) shows the area-averaged mean internal pressure, $\langle {}^i p \rangle$ which increases with ϵ and decreases with β. Finally, figure 3(*e*) shows the relationship between the circumferential stretch ratio, λ, and the parameters ϵ and β. As ϵ increases so does λ for all β, but the strain $(\lambda - 1)$ never exceeds 30% for the range of ϵ examined. Note, however, that λ is much more sensitive to β than is D.

5. Apparent viscosity of a dilute suspension

To conclude this paper we will use our solution for the steady shear flow about a flexible cylindrical capsule to calculate the apparent viscosity of a dilute suspension of such particles. The suspension must, of course, be dilute because our solution refers to an isolated capsule in an unbounded flow, and interactions with neighbouring particles which would be important in a concentrated solution are not accounted for. The relations between the bulk flow properties of a suspension and the micromechanical properties of suspended particles are of great interest in suspension rheology. In spite of the somewhat artificial two-dimensional nature of our model, these calculations can offer some interesting qualitative insights into this important problem, particularly because our results are not restricted to small deformations.

The details of the calculation of the apparent viscosity, ${}^a\hat{\mu}$, can be found in (Rao 1991). The apparent viscosity is defined as the ratio of the mean shear stress to the mean velocity gradient in a steady shear flow of the suspension (Lightfoot 1974). Following common rheological practice we express the apparent viscosity in terms of a parameter called the 'intrinsic viscosity' $[\mu]$, defined by

$$[\mu] = ({}^a\hat{\mu} - {}^e\hat{\mu})/({}^e\hat{\mu}c), \tag{46}$$

where c is the volume fraction of the suspended particles (per unit width, in our two-dimensional problem). The calculations in Rao (1991) show that

$$[\mu] = 2\{1 - \tfrac{1}{8}(\tfrac{1}{6}\epsilon)^{\frac{2}{3}}(1 + 12\beta) + O(\epsilon^{\frac{4}{3}})\} \tag{47}$$

for a suspension of flexible cylinders and, further, that if the cylinders are rigid and of circular cross-section then $[\mu] = 2$. These results can be compared with the well-known fact that for dilute suspensions of rigid *spheres* $[\mu] = 2.5$ (Lightfoot 1974).

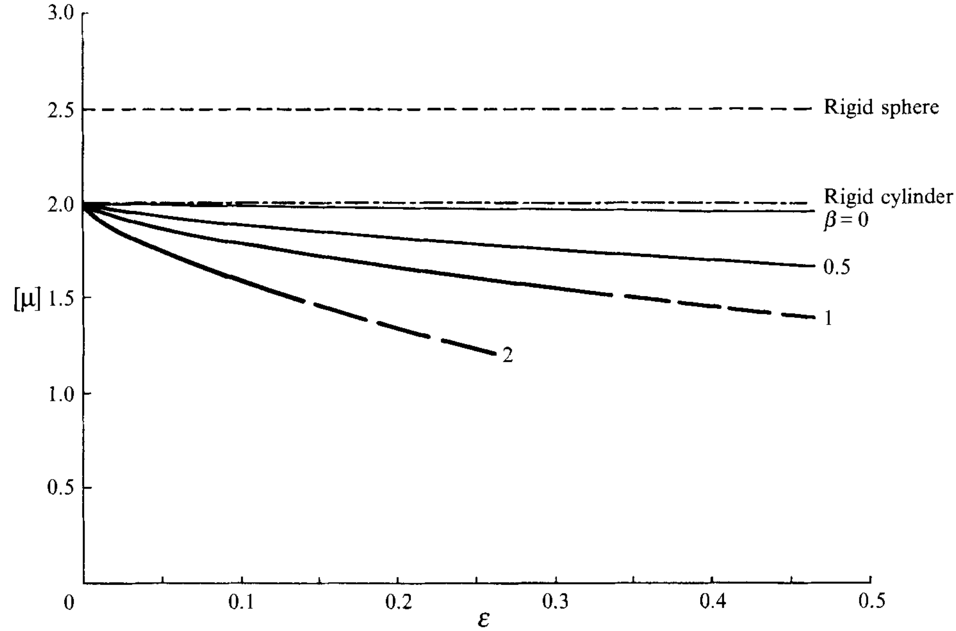

FIGURE 4. Dependence of the intrinsic viscosity $[\mu]$ on ϵ and β. For fixed membrane and fluid properties the apparent viscosity decreases with shear rate. For fixed shear rate, membrane stiffness, and external fluid viscosity, the apparent viscosity decreases with increasing internal fluid viscosity. Dashed lines indicate probable limits of accuracy of (47).

In figure 4 we have summarized the dependence of $[\mu]$ on various material and flow parameters, according to (47). Note that if both ϵ and β are large, then the deviation of $[\mu]$ from 2 is so great that the accuracy of the calculation is questionable (that is, the unknown terms of $O(\epsilon^{\frac{4}{3}} = \delta^4)$ may be important). It is evident that $[\mu]$, and therefore, the apparent viscosity, for a suspension of flexible particles, is lower than that for rigid cylindrical particles which, in turn, is lower than that for rigid spherical particles. For fixed β, $[\mu]$ decreases as the $\frac{2}{3}$ power of the dimensionless shear rate, ϵ. The behaviour of $[\mu]$ with respect to the viscosity ratio, β, is counter-intuitive. As β increases, which could occur by increasing the internal viscosity with fixed external viscosity, $[\mu]$ decreases for any given ϵ. The significance of this result is discussed in the next section.

6. Discussion

This analysis of elastic capsule deformations in shear flows presents some interesting features both from an intrinsic mathematical viewpoint and in relation to experiments on erythrocytes. Although the solution involves the analysis of a regular (non-singular) perturbation problem, this case of an initially unpressurized two-dimensional capsule without bending rigidity is substantially more difficult than when the capsule is pressurized (Zahalak *et al.* 1987). In the latter case the perturbation series is completely straightforward with all variables of the same order, $O(\epsilon)$. This is also the case for a membrane whose elastic resistance is dominated by bending rigidity, as shown by Rao (1991). In the present problem, however, one must assume different orders of magnitude for different variables ((18)–(25)) in order to construct a consistent recursive algorithm. Apparently these difficulties do not arise in the corresponding three-dimensional problems (Barthès-Biesel & Rallison 1981; Barthès-Biesel & Sgaier 1985)

Increase in → Change in ↓	$\dot{\gamma}$	${}^{e}\hat{\mu}$	${}^{i}\hat{\mu}$	$\hat{G}$
D	↑	↑	↓	↓
θ_{max}	↓	→	↓	↑
$\hat{\omega}$	↑	→	↓	→
$[\mu]$	↓	→	↓	↑

TABLE 3. Summary of interactions at steady state; → denotes approximately constant.

where simple perturbation series are assumed with all variables of the same order of magnitude in the dimensionless shear rate. The three-dimensional problems, however, involve much more difficult calculations and published analyses are confined almost invariably to the leading term of the series or, in rare cases, two terms (Barthès-Biesel 1980). One consequence of the fact that series of several different orders enter our problem is that to determine completely all the terms up to a given order m, the problems of orders $m+1$, $m+2$ and $m+3$ must be considered; in particular, the first-order shape function $f^{(1)}$ is not determined completely until the fourth-order problem is examined.

In the face of this complexity computer assistance is essential and we have solved this problem employing the symbol-manipulation programs, REDUCE, and the techniques which we developed previously in treating the problem of an initially pressurized inextensible membrane. Complete listings of the programs used in the solution of the current problem, both with and without bending rigidity are available in Rao (1990). REDUCE both formulated and solved the perturbation problems of succeeding orders and enabled us to carry the analysis to the problem of $O(\delta^7)$, a sufficiently high order to accommodate quite large deformation (up to $D \approx 3$).

While the formulation of the problem we have posed and solved, (4)–(17), is for the most part straightforward within the context of two-dimensional, incompressible, creeping flow, the 'global equilibrium condition', (14), merits some additional comments and motivation. For a long cylindrical membrane closed by end-caps the correct equilibrium condition would be

$$\oint_{\Gamma} Q\,\mathrm{d}s = \iint_{\mathscr{A}} {}^{i}p\,\mathrm{d}\mathscr{A} - \iint_{\mathscr{A}} {}^{e}p\,\mathrm{d}\mathscr{A}. \tag{48}$$

Equation (14) neglects the last term – that is, the axial force due to the external pressure. It is certainly expedient to neglect this term as our two-dimensional formulation gives no information about the pressure distribution on the hypothetical end-caps, but one can put forward the following physical arguments to rationalize this assumption. First, dimensional considerations suggest that ${}^{e}\hat{p}$, the physical external pressure, should scale like $({}^{e}\hat{\mu}\hat{v}_0/\hat{a})$, where $\hat{v}_0$ is a characteristic velocity. In the problem under consideration the obvious choice for $\hat{v}_0$ is $\hat{a}\dot{\gamma}$, so that the external pressure is of the order of ${}^{e}\hat{\mu}\dot{\gamma}$. We scale the internal pressure as ${}^{i}\hat{p} = (\hat{G}/\hat{a}){}^{i}p$, so the condition ${}^{i}\hat{p} \gg {}^{e}\hat{p}$ would be satisfied if $(\hat{G}/\hat{a}){}^{i}p \gg {}^{e}\hat{\mu}\dot{\gamma}{}^{e}\tilde{p}$, where ${}^{e}\tilde{p} = ({}^{e}\hat{p}/{}^{e}\hat{\mu}\dot{\gamma})$ is a dimensionless external pressure which is presumably $O(1)$ on the end-caps. Thus, the external pressure could be neglected with respect to the internal pressure if

$${}^{i}p \gg \hat{a}\dot{\gamma}\,{}^{e}\hat{\mu}/\hat{G} = \epsilon. \tag{49}$$

An examination of figure 3(*d*) confirms *a posteriori* that ${}^{i}p$ is generally greater than ϵ, at least on the average, which tends to support our assumption.

To require ${}^{i}p \gg {}^{e}\hat{p}$ (that is, that the internal pressure exceed significantly the external pressure at every point on the end-caps) is actually too stringent. All that is required for (14) to hold is that the integral of the external pressure – the external force – should be negligible. The physical nature of our problem suggests that the external pressure would be spatially periodic, with regions of positive and negative pressure which tend to cancel on integration. As a specific example consider a rigid sphere rotating in a shear flow (Lightfoot 1974). In this particular case it is easy to show that the external pressure indeed has the postulated spatially periodic character, and that the net axial pressure force on each hemisphere is exactly zero (when, as in this paper, the pressure is measured with respect to the remote pressure at infinity). The internal pressure, on the other hand, has a non-zero mean as shown in figure 3(*d*), which increases with capsule deformation. While these arguments obviously do not constitute a mathematical proof, we believe that they indicate that our particular formulation of the global equilibrium condition is not physically unreasonable.

In attempting to apply our results to the behaviour of erythrocytes in the Rheoscope it must be borne in mind that the model we have analysed suffers from at least two deficiencies for this purpose: (i) the initial unstressed, unpressurized cross-section of our capsule is circular, whereas that of the erythrocyte is 'dumbbell-shaped', and (ii) our global equilibrium condition, (14), is only a rough approximation of the true lateral constraint provided by a compact closed three-dimensional capsule. Nevertheless the results may describe at least qualitatively what occurs in Rheoscope experiments.

In §4 we presented the behaviour of several non-dimensional variables associated with the local and global aspects of deforming cylindrical capsules under steady-state conditions. We shall now examine those results in terms of some physical (dimensional) variables. These results will then be compared with theoretical and experimental results for red blood cells, liquid drops and flexible spherical capsules. We can interpret and summarize the interaction of the material parameters, shear rate, and capsule response as shown in table 3.

While most of the effects indicated in table 3 are intuitive, some are not, as in the case of a decrease in $[\mu]$ with an increase in ${}^{i}\hat{\mu}$, or the absence of any noticeable change in $\hat{\omega}$ with an increase in ${}^{e}\hat{\mu}$. The paradoxical effect of internal viscosity on intrinsic viscosity can be rationalized to some extent. As the intrinsic viscosity is a measure of the *additional* dissipation produced by the suspended particles, one might suppose that it increases with the degree of disturbance of the basic shear flow. One might suppose further that particles which are elongated and aligned with the flow will produce less disturbance, and therefore a lower $[\mu]$. From figure 3 we can see that an increase in the internal viscosity produces only a small decrease in the membrane deformation, but the angle of orientation of the membrane with respect to the flow direction decreases significantly. Thus, membranes with higher internal viscosity tend to create less disturbance of the surrounding flow, and consequently yield lower apparent viscosity and $[\mu]$. We should emphasize, however, that our solution is not valid at high values of β, where one would expect $[\mu]$ to approach the value of 2 appropriate for rigid spheres. The lack of any noticeable increase in the intrinsic viscosity with an increase in the external fluid viscosity may also be surprising. This seems to be a balance of two competing effects. As the external viscosity increases at constant shear rate, ϵ increases, thus tending to decrease $[\mu]$. But then β also decreases if the internal viscosity is constant, tending to increase $[\mu]$. Note that the apparent viscosity will still increase because it is proportional to the viscosity of the suspending medium. These considerations suggest that the influence of capsule deformability on suspension viscosity is subtle, and may not be predictable on a purely intuitive basis. Another interaction listed in table 3 that needs explanation is the absence of any effect of the suspending fluid viscosity on the tank-treading frequency. There exists somewhat conflicting experimental evidence on this point with regard to red blood cells. While Sutera *et al.* (1983) found that the $\hat{\omega}$ increases weakly with external viscosity, Fischer *et al.* (1978) found no effect. These theoretical and experimental results make the point that we should not expect intuitively an increase in the $\hat{\omega}$ with an increase in ${}^{e}\hat{\mu}$ merely because the applied shear stress increases.

It is instructive to compare our results to previously published analyses of flexible particles in shear flow, with the caveat that these analyses were three-dimensional and limited to small deformations. Cox (1969) and Torza, Cox & Mason (1972) obtained a two-term solution to the problem of a liquid drop in a shear flow. Their analysis indicates the following. For $\beta = O(1)$ and $k = P/(\hat{a}\,{}^{e}\hat{\mu}\dot{\gamma}) \rightarrow \infty$ (implying low shear rates), as the applied shear rate, $\dot{\gamma}$, increases, the deformation, D, and the $\hat{\omega}$ increase, and the angle of orientation θ_{max} decreases. For the same conditions on β and k, as β increases the deformation and the angle of orientation decrease, but the tank-treading frequency increases. In our analysis of a cylindrical fluid-filled membrane we find similar effects of viscosity ratio and applied shear rate on membrane deformation and the angle of orientation. But the dependence of tank-treading frequency on the viscosity ratio for a liquid drop is inverse of that seen in our membrane model (figure 3*c*). The liquid-drop results also state that, for $k = O(1)$ and $\beta \rightarrow \infty$, as the applied shear rate $\dot{\gamma}$ increases the deformation approaches $1 + 5/(4\beta)$, the angle of orientation with respect to the direction of flow remains at zero while the tank-treading frequency increases. We cannot compare directly these results to those we have reported in this paper, because our expansions are not valid for large β. However, Rao (1991) has performed a brief auxiliary analysis which provides, up to the second term, the steady-state solution for an elastic cylindrical membrane with a uniform stretch ratio containing a highly viscous Newtonian fluid placed in a slow viscous shear flow of another Newtonian fluid of low viscosity. According to that analysis, as $\beta \rightarrow \infty$, $D = r_{max}/r_{min} \rightarrow 1 + 2\beta^{-1}$, $\theta_{max} \rightarrow 0$ and ${}^{m}u_s \rightarrow \frac{1}{2}\epsilon(1 - 2\beta^{-1})$, where θ_{max} is the orientation with respect to the direction of flow, and ${}^{m}u_s$ is the tangential membrane velocity. Thus, for the two-dimensional elastic membrane in the limit of $\beta \rightarrow \infty$, as the applied shear rate (or ϵ) increases D and θ_{max} are unaffected to first-order while $\hat{\omega}$ increases (see (43)). However, as β decreases the deformation increases and the tank-treading frequency decreases. This is exactly the same behaviour as seen in a liquid drop. In summary, a liquid droplet and a flexible cylindrical membrane in general behave similarly in response to changes in the applied shear rate and the viscosity ratio. An exception is the variation of $\hat{\omega}$ for small to moderate viscosity ratios ($\beta = O(1)$) and low shear rates ($k \rightarrow \infty$); under these conditions the $\hat{\omega}$ of a liquid drop increases with β, whereas the opposite is true for the flexible cylindrical capsule.

Barthès-Biesel & Sgaier (1985) studied a viscoelastic spherical capsule in a shear flow. Their results agree with the findings of the present analysis for a cylindrical membrane with regard to the effect of membrane stiffness and applied shear rate on the membrane deformation and orientation: as the membrane stiffness increases the deformation decreases and the angle of inclination with respect to the direction of the flow increases. The effects of variations in shear rate are the opposite of those in membrane stiffness, as is the case in our analysis. But no effect of the membrane viscosity under steady-state conditions is predicted by our analysis because the membrane strain is uniform, which results in a zero viscous force irrespective of the membrane shear viscosity. Barthès-Biesel & Sgaier (1985) found an effect of membrane viscosity similar to that of the membrane stiffness on the membrane deformation. Their analysis also predicted an oscillatory motion at high membrane or internal fluid viscosities. No such behaviour is seen in the present analysis for a cylindrical membrane, possibly because this analysis is not valid for sufficiently large viscosity ratios.

There appears to be another striking difference between the results for a spherical and a cylindrical membrane. According to Barthès-Biesel & Sgaier (1985), in the case of a spherical membrane all field variables are of the same order in $\epsilon = k^{-1}$ as this parameter approaches zero. This is not the case for our unpressurized cylinder. Specifically, for a sphere the deformation (change in the membrane shape) is $O(\epsilon)$ and the velocity and pressure are also $O(\epsilon)$. But for a cylindrical membrane we find that the deformation is $O(\epsilon^{\frac{1}{3}})$ and the velocity and external pressure are $O(\epsilon)$. The reason for this discrepancy is not clear, but it is well known that a change from a three-dimensional spherical geometry to a two-dimensional cylindrical geometry may cause profound differences in the solutions of seemingly similar problems. (A famous example is Stokes' Paradox in the steady creeping flow of a viscous fluid around a fixed cylinder versus a fixed sphere (Langlois 1964).) The elimination of the viscous terms from the constitutive equation cannot be the cause of the disparity between the cylindrical and the spherical models because this disparity exists even for a purely elastic sphere (Barthès-Biesel 1980).

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